Stratified p-Center Problem with Capacity Constraints and Failure Foresight

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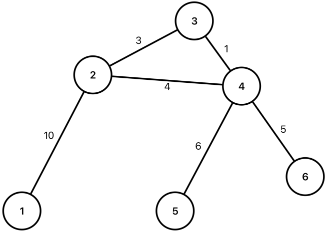
Problem description: p-center problem

NP-Hard Optimization problem

Set $G = (N, E)$ a graph where :

- N is the set of nodes (cities)
- E is the set of weighted edges (roads)
- d_{ij} is the minimum distance between cities i and j

Exactly p centers (schools) to be placed on nodes.



p-center problem

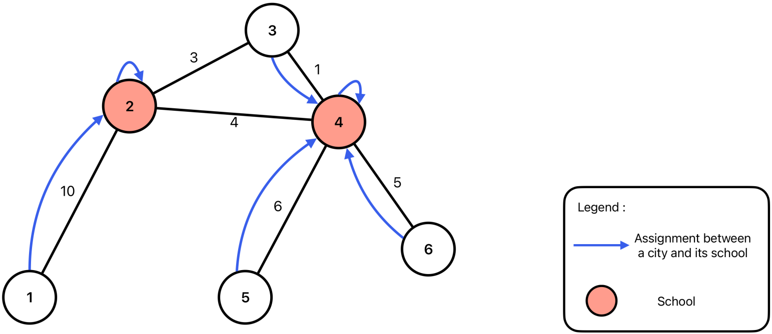
Find subset $P \subset N$ such that $|P| = p$, by minimizing the maximum distance between cities and their nearest school:

$$\min_{\substack{P \subset N \\ |P| = p}} \max_{i \in N} \min_{j \in P} d_{ij}$$

Problem description: *p*-center problem

Description:

- 6 cities
- 2 schools to build
- Each city is assigned to its closest school



Variants

There are several variants of the problem:

- Adding p centers with q existing centers
- Adding capacity constraints
- Taking uncertain parameters into account
- Stratifying the problem
- Managing centers failures
- ...

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- ...

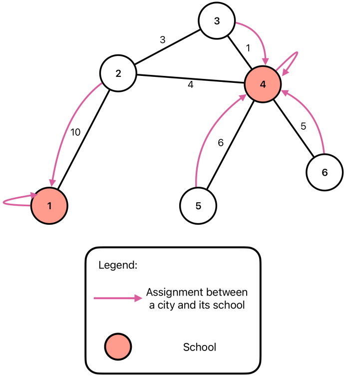
Combination of 3 variants

Demand: number of pupils to be assigned to schools

Capacity: maximum number of pupils that a school can accept.

Node	1	2	3	4	5	6
Demand	3	5	1	3	2	3
Capacity	9	6	2	9	4	4

Table: Graph capacities and demands



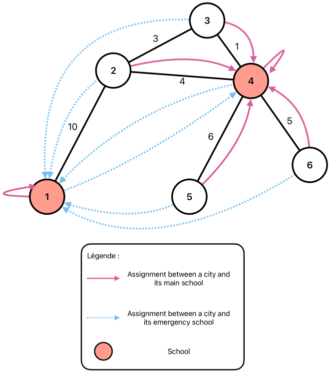
Samir Khuller and Yoram J.Sussmann. The Capacitated K-Center Problem, SIAM Journal on Discrete Mathematics, vol.13, no.3, pp. 403–418, Jan. 2000

Added: A school may be closed

- All pupils assigned to the closed school must be redirected to another school (emergency school)

Node	1	2	3	4	5	6
Demand	3	5	1	3	2	3
Capacity	17	6	2	18	4	4

Table: Graph capacities and demands



The emergency school is the second closest school available according to capacity

Inmaculada Espejo, Alfredo Marín, and Antonio M. Rodríguez-Chía. Capacitated *p*-center problem with failure foresight, European Journal of Operational Research, vol. 247, no. 1, pp. 229–244, Nov. 2015
Yolanda Hinojosa, Alfredo Marín, and Justo Puerto. Dynamically second-preferred p-center problem, European Journal of Operational Research, vol. 307, no. 1, pp. 33-47, 2023

Problem description: Stratification

Added:

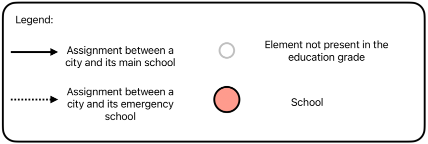
- We have a set of services (educational stages):

$$S = \{\text{1st grade, 2nd grade, ...}\}$$

- Each city asks for different educational grades with different quantity of pupils
- Each school provides a subset of educational grades with distinct capacities
- The grades requested or provided are not necessarily all available in the same city

Maria Albareda-Sambola, Luisa I. Martínez-Merino, and Antonio M.Rodríguez-Chía. The stratified p-center problem, Computers & Operations Research, vol. 108, pp. 213–225, Aug. 2019

Plain lines: Assignment to the main school
Dotted lines: Assignment to the emergency school



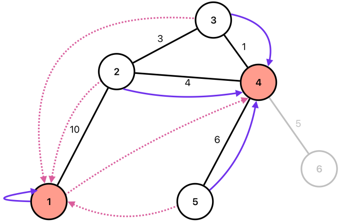


Figure: 1st grade

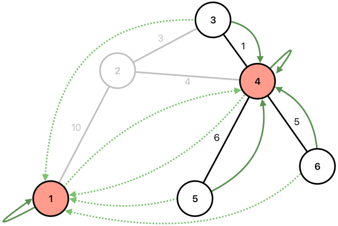


Figure: 2nd grade

Difficulty: Find a feasible solution common to each grade

Data

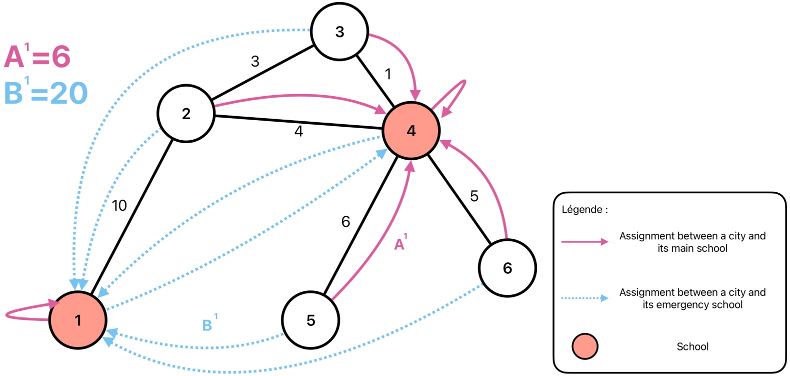
In this combined problem, we have:

- N the set of cities and potential school locations
- E the set of weighted edges
- p schools to be placed
- S: the set of educational grades
- d_{ij}: the minimum distance between cities i and j, $i, j \in N$
- q_{is}: number of pupils to be assigned for the class $s \in S$ in city $i \in N$
- u_{js}: number of pupils of the class $s \in S$ that the school $j \in N$ can accept

Evaluation

- $A^s = \max_{i \in N}\{d_{ik} \mid\ k$ is the main school of city i for grade $s\}$, the maximum distance between cities and their main school for grade $s \in S$
- $B^s = \max_{i \in N}\{d_{ik} \mid\ k$ is the emergency school of city i for grade $s\}$, the maximum distance between cities and their emergency school for grade $s \in S$

Example of A and B with one stratum



Evaluation

- $A^s = \max_{i \in N}\{d_{ik} \mid k$ is the main school of city i for grade $s\}$, the maximum distance between cities and their main school for grade $s \in S$
- $B^s = \max_{i \in N}\{d_{ik} \mid k$ is the emergency school of city i for grade $s\}$, the maximum distance between cities and their emergency school for grade $s \in S$

In the literature, the objective is typically to minimize B^s.

Our questions:

- Is this objective still valid and sufficient for our problem context?
- If we introduce A^s into the evaluation, can it bring additional relevance?
- If so, what is its impact on the structures and performances?

Compared objective functions

- $f = \sum_{s \in S} B^s$: inspired by the literature. Minimize the maximum distance between cities and their emergency school for each grade.
- $g = \sum_{s \in S}(A^s + B^s)$: our contribution. Minimize the maximum distance between cities and their main school but also emergency school, for each grade.
- $h = \sum_{s \in S} A^s$: is used only for bounds. Minimize the maximum distance between cities and their main school for each grade.

is f relevant for our combined problem?

Problem model

$$\text{Min } F \tag{1}$$

$s.c$

$$\sum_{j \in N} y_j = p \tag{2}$$

$$\sum_{\substack{j \in N \\ s \in C_j}} x_{ijs} = 1 \qquad \forall i \in N,\ \forall s \in S_i \tag{3}$$

$$\sum_{\substack{j \in N \\ s \in C_j}} w_{ijs} = 1 \qquad \forall i \in N,\ \forall s \in S_i \tag{4}$$

$$x_{ijs} \leq y_j \qquad \forall i, j \in N,\ s \in S_i \cap C_j \tag{5}$$

$$w_{ijs} \leq y_j \qquad \forall i, j \in N,\ s \in S_i \cap C_j \tag{6}$$

$$x_{ijs} + w_{ijs} \leq 1 \qquad \forall i, j \in N,\ s \in S_i \cap C_j \tag{7}$$

$$\sum_{\substack{j \in N \\ s \in C_j}} d_{ij} \cdot x_{ijs} \leq A^s \qquad \forall i \in N,\ s \in S_i \tag{8}$$

$$\sum_{\substack{j \in N \\ s \in C_j}} d_{ij} \cdot w_{ijs} \leq B^s \qquad \forall i \in N,\ s \in S_i \tag{9}$$

$$\sum_{\substack{j \in N \\ s \in C_j}} d_{ij} \cdot x_{ijs} \leq \sum_{\substack{j \in N \\ s \in C_j}} d_{ij} \cdot w_{ijs} \qquad \forall i \in N,\ s \in S_i \tag{10}$$

$$\sum_{\substack{i \in N \\ s \in S_i}} q_{is} \cdot x_{ijs} + \sum_{\substack{i \in N \\ s \in S_i}} q_{is} \cdot w_{ijs} \leq u_{js} \cdot y_{js} \qquad \forall j \in N,\ s \in C_j \tag{11}$$

$$x_{ijs} \in \{0, 1\} \qquad \forall i, j \in N,\ s \in S_i \cap C_j \tag{12}$$

$$w_{ijs} \in \{0, 1\} \qquad \forall i, j \in N,\ s \in S_i \cap C_j \tag{13}$$

$$y_j \in \{0, 1\} \qquad \forall j \in N \tag{14}$$

$$A^s, B^s \geq 0 \qquad \forall s \in S \tag{15}$$

The number of constraints and variables is of the order $O(|S| \cdot |N|^2)$.

Experimental protocol

For each objective function (f, g and h), we analyze:

- The structure of optimal or suboptimal solutions according to two criteria
- Performance in terms of time

We tested our benchmarks on IBM ILOG Cplex 22.1.1 solver with a time limit of 3600 seconds.

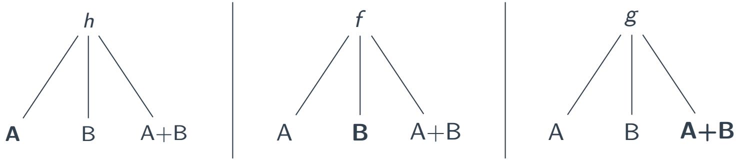
Instances family

The instances are drawn from the literature and have been enriched.

Family	N	p	S	Number of instances
Pmed	[100, 800]	$[5, \frac{N}{3}]$	$\{3, 5, 10, 15, 20\}$	54
Beasley	$\{100, 150\}$	5 when $N = 50$, 10 when $N = 100$	$\{3, 5, 10, 15, 20\}$	99
Galvao	$\{100, 150\}$	$\{5, 15\}$	$\{3, 5, 10, 15, 20\}$	31
Lorena	[100, 402]	[10, 40]	$\{3, 5, 10, 15, 20\}$	7
Random	$\{20, 25, 30, 35,$ $40, 45, 50\}$	$\{\frac{N}{10}, \frac{N}{5}, \frac{N}{3}\}$	$\{\frac{N}{10}, \frac{N}{5}\}$	42

Criteria for comparing objective functions

For each instance resolved, we have:



with: $A = \sum_{s \in S} A^s$ $\quad B = \sum_{s \in S} B^s$ $\quad A + B = \sum_{s \in S} (A^s + B^s)$

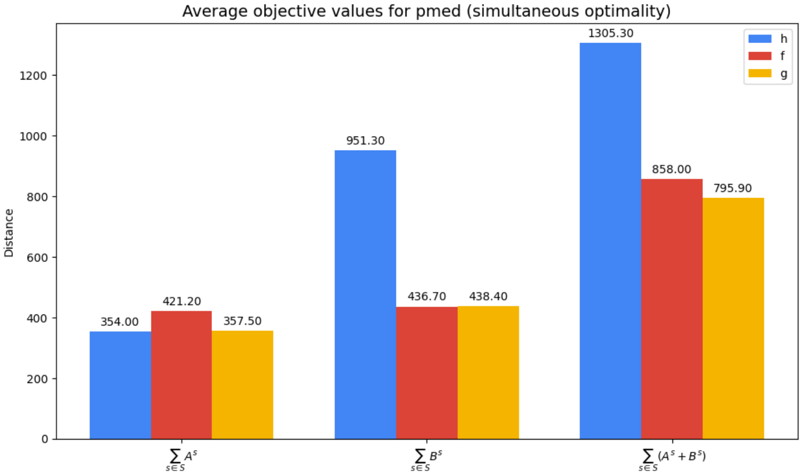
We compare between these objective functions:

- The mean of A, B and $A + B$ values for each instance family
- The number of occurrences where f (resp. g and h) obtains a better or equivalent value of A (resp. B and $A + B$) for each instance family.

Structural analyses: Objective values in case of simultaneous optimality

- Y-axis: Average of objective values in case of simultaneous optimality

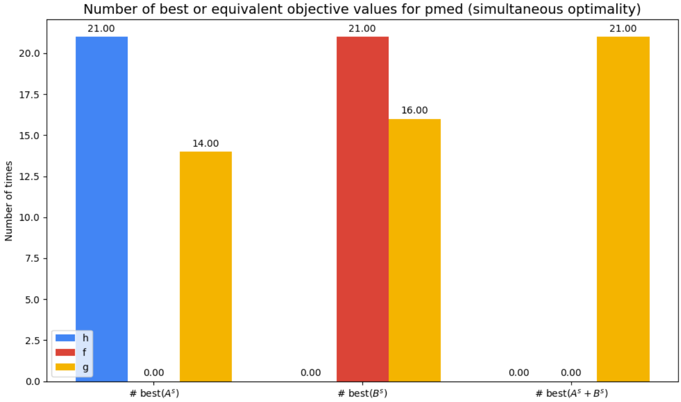
$$h = \sum_{s \in S} A^s \qquad f = \sum_{s \in S} B^s \qquad g = \sum_{s \in S} (A^s + B^s)$$



Structural analyses: Number of times where an objective function gets the best objective values in case of simultaneous optimality

- Y-axis: Number of cases where one objective function produces a better or equivalent result than the others, in case of simultaneous optimality.

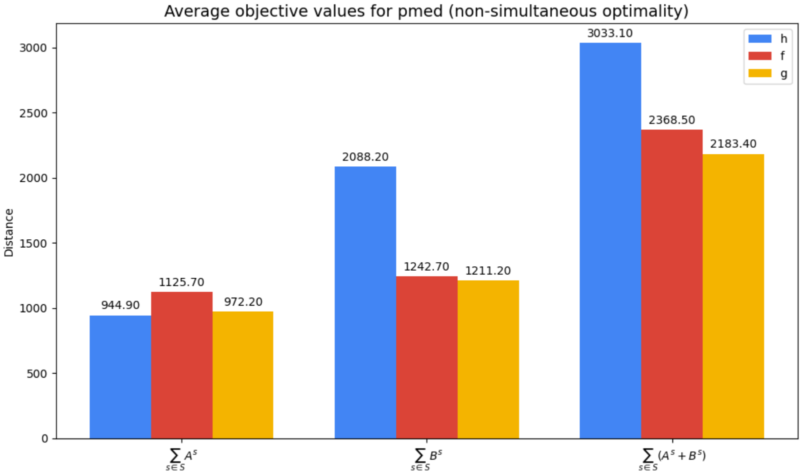
$$h = \sum_{s \in S} A^s \qquad f = \sum_{s \in S} B^s \qquad g = \sum_{s \in S} (A^s + B^s)$$



Structural analyses: Objective values in case of non-simultaneous optimality

- Y-axis: Average of objective values in case of non-simultaneous optimality

$$h = \sum_{s \in S} A^s \qquad f = \sum_{s \in S} B^s \qquad g = \sum_{s \in S} (A^s + B^s)$$



Structural analyses: Number of times where an objective function gets the best objective values in case of non-simultaneous optimality

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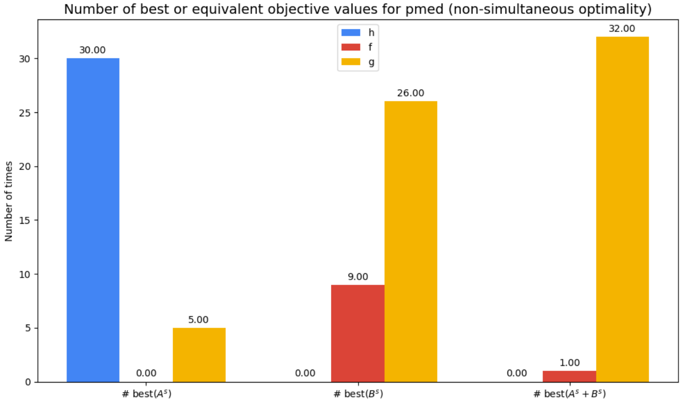


Further comparison criteria

In the case of simultaneous optimality by f, g and h, for each instance we define:

- $\mathcal{A}^s = \frac{1}{|N|} \sum_{i \in N} d_{i,P_s(i)}$: average distance between each city i and its assigned **main school** for stratum $s \in S$.
- $\mathcal{B}^s = \frac{1}{|N|} \sum_{i \in N} d_{i,E_s(i)}$: average distance between each city i and its assigned **emergency school** for stratum $s \in S$.

We then define the following sets:

$$M_{\mathcal{A}} = \{\mathcal{A}^s \mid s \in S\}, \qquad M_{\mathcal{B}} = \{\mathcal{B}^s \mid s \in S\}$$

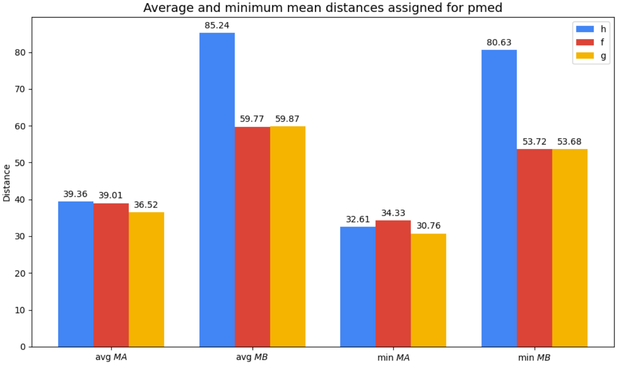
For each objective function, we compare the **minimum** and **mean** values over the sets $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$:

- $\min M_{\mathcal{A}} = \min_{s \in S} \mathcal{A}^s$
- $\operatorname{mean} M_{\mathcal{A}} = \frac{1}{|S|} \sum_{s \in S} \mathcal{A}^s$
- $\min M_{\mathcal{B}} = \min_{s \in S} \mathcal{B}^s$
- $\operatorname{mean} M_{\mathcal{B}} = \frac{1}{|S|} \sum_{s \in S} \mathcal{B}^s$

Structural analyses: Average Assignment Distances

- Y-axis : Value of average and minimum assigned mean distances for each function.

$$h = \sum_{s \in S} A^s \qquad f = \sum_{s \in S} B^s \qquad g = \sum_{s \in S} (A^s + B^s)$$

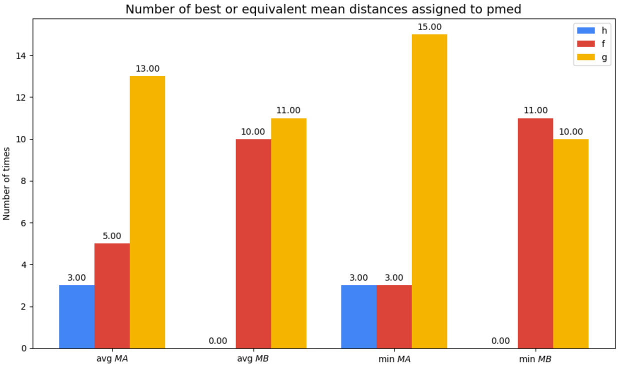


Smaller values are better.

Structural analyses: Average Assignment Distances

- Y-axis: Number of cases in which an objective function has an average or minimum of assigned mean distances that is less than or equal to other functions.

$$min\ h = \sum_{s \in S} A^s \qquad min\ f = \sum_{s \in S} B^s \qquad min\ g = \sum_{s \in S} (A^s + B^s)$$



Summary

In terms of objective values:

- In case of simultaneous optimality:
  - f gets better B **(**g **is** 2.66% **higher and** h **is** 85.05% **higher)**
  - g gets better $A + B$ **(**f **is** 6.88% **higher and** h **is** 44.97% **higher)**
  - h gets better A **(**g **is** 3.27% **higher and** f **is** 23.72% **higher)**
- In case of non-simultaneous optimality:
  - Objective function g gets better B and $A + B$ values
  - Objective function h gets better A values

In terms of structures:

- The g function obtains better minimum and mean values on $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$
- The g function more often achieves equal or better $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$ (except for min $M_{\mathcal{B}}$)

Results in terms of time

- $T(s)$ being the time in seconds
- *Node* being the number of nodes developped by CPLEX

$$h = \sum_{s \in S} A^s \qquad f = \sum_{s \in S} B^s \qquad g = \sum_{s \in S} (A^s + B^s)$$

Instances	F	$T_{avg}(s)$	$T_{min}(s)$	$T_{max}(s)$	$Node_{avg}$
Pmed^{sim}	f	1196.1	14.6	3516.5	836.3
	g	664.6	**8.5**	2890.0	461.4
	h	**467.1**	10.2	**1913.4**	34.3
Beasley^{sim}	f	910.5	120.8	3442.9	16757.3
	g	166.2	11.2	487.8	5244.0
	h	**43.3**	**4.5**	**209.3**	592.7
Random	f	135.0	0.1	3102.1	4429.3
	g	48.9	0.1	783.4	2525.2
	h	**9.7**	0.1	**163.9**	746.0

Table: Time results for simultaneous optimality

Same observations in the case of non-simultaneous optimality

Conclusion

- A new problem combining 3 known variants:
  - Capacity constraints
  - Failure foresights
  - Stratifying the problem

- Comparison of three objective functions:
  - $h = \sum_{s \in S} A^s$
  - $f = \sum_{s \in S} B^s$ (inspired by literature)
  - $g = \sum_{s \in S} (A^s + B^s)$

- According to our analyses, the g function:
  - obtains solutions with better or equivalent structure
  - performs better in terms of time compared to f (but less than h)

is f relevant for our problem?

No, it is not!

Perspectives

We would like to:

- In modeling terms:
  - Add weights to each service to prioritise some of them or not
  - Generalize the 3 objective functions

- In terms of resolution:
  - Break down the problem stratum by stratum
  - Introduce valid inequalities
  - Develop a heuristic or a local search to solve the problem in an approximate way.

Thanks for listening!

Do you have any questions?

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